Topics in the Philosophy and Foundations of Mathematics

Lecture 1: Logicism and Formalism

Claudio Ternullo

12 May 2022

Universitat de Barcelona

Basic Info

- Course length will be 8 weeks: 7 lectures + 1 exam session
- The course will be held on Thursday every week except on *Week 4*, when it'll be held on Friday (3rd of June)
- Time is 11-13am
- A 'minimal' bibliography has already been provided in the course description, but additional items will be discussed while the course is running
- The last session on *Week 8* will be the final exam
- The language used will be English, unless all attendants' mother tongue is Italian

Course Structure

- **Lecture 1** (12 May). *Logicism and Formalism.*
- **Lecture 2** (19 May). *Intuitionism, Deductivism and Hilbert's Programme.*
- **Lecture 3** (26 May). *The Incompleteness Theorems.*
- **Lecture 4** (3 June). *The Birth of Set Theory (Cantorian Set Theory)*
- **Lecture 5** (9 June). *The Set-theoretic Axioms. Independence and Open Problems.*
- **Lecture 6** (16 June). *Alternative Conceptions of the Infinite (Non-Standard Theories).*
- **Lecture 7** (23 June). *The Question of Realism*
- **Lecture 8** (30 June). Exam Session.

Exam Session

- Only for doctoral students.
- It may consist in: (1) a short presentation or essay on an assigned topic; (2) a final questionnaire
- Options will have to be discussed with the Instructor beforehand.

Minimal Bibliography

1. Franzen, T. *Gödel's Theorem. An incomplete guide to its use and abuse*, Routledge, London, 2005
2. Hamkins, J. D. *Lectures on the philosophy of mathematics*, MIT Press, Cambridge (MA), 2021
3. Linnebo, Ø. *Philosophy of mathematics*, Princeton University Press, Princeton (NJ), 2017
4. Lolli, G. *Filosofia della matematica. L'eredità del Novecento*, Il Mulino, 2002
5. Panza, M. – Sereni, A. *Il problema di Platone. Un'introduzione storica alla filosofia della matematica*, Carocci editore, Roma, 2010
6. Shapiro, S. *Thinking about mathematics: the philosophy of mathematics*, Oxford University Press, Oxford, 2000
7. - (ed.) *Oxford Handbook of Philosophy of Mathematics and Logic*, Oxford University Press, Oxford, 2005
8. Ternullo, C.-Fano, V. *L'infinito. Filosofia, matematica, fisica*, Carocci editore, Roma, 2021

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- Frege's two main contributions to the foundations and philosophy of mathematics may be briefly described as follows:

1. The introduction and clarification of the notion of *formal system* (for which, see [Frege, 1879])
2. The attempt to reduce relevant parts of mathematics, mostly, *arithmetic*, to *logic*

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- Kant thought that mathematical propositions were *synthetic a priori*
- Frege stretches the notion of *analytic*, which he takes to mean: 'derivable from (purely) logical principles/laws'

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- Fundamental to number ascriptions is Hume's Principle:

Hume's Principle (HP)

The *number of* F*s* is equal to the *number of* G*s* if and only if F is *equinumerous with* G. In symbols:

$$\#x\ F(x) = \#x\ G(x) \leftrightarrow F \approx G$$

Peano's (and Frege's) Arithmetic

Second-Order Peano(-Dedekind) Arithmetic (PA_2, cf. [Peano, 1889], [Peano, 1891], [Dedekind, 1888])

1. $N(0)$.
2. $N(x) \wedge P(y, x) \rightarrow N(y)$.
3. $N(x) \wedge (P(x, y) \wedge P(x, y')) \rightarrow y = y'$.
4. $N(y) \wedge (P(x, y) \wedge P(x', y)) \rightarrow x = x'$.
5. $N(x) \rightarrow \exists y P(x, y)$.
6. (Induction).
   $(\forall F)(F(0) \wedge (\forall x)(\forall y)(((N(x) \wedge F(x) \wedge P(x, y)) \rightarrow F(y)) \rightarrow (\forall x)(N(x) \rightarrow F(x))$.

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In order to *reduce* PA_2 to logic, Frege needs to show that: '$N(x)$' and '$P(x, y)$', '0' are *reducible* to logic.

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(SOL)+(HP) $\vdash PA_2$.

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Fix: define numbers as *equivalence classes* of *equinumerous concepts* as follows:

$$[0] = \{F : F \approx \#(x \neq x) \approx 0\}$$
$$[1] = \{F : F \approx 0\}$$
$$[2] = \{F : F \approx 0 \vee F \approx 1\}$$
$$[3] = \{F : F \approx 0 \vee F \approx 1 \vee \approx 2\}$$

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This is known as Basic Law V (BLV), and stands out as the central pillar of Frege's logicist system in [Frege, 1903]. As shown by Bertrand Russell while Frege's *Grundgesetze* were in print, BLV is prone to paradox:

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Take $(x : F(x)) = \{x : x \notin x\}$. Then, there exists the class of all classes which do not belong to themselves, R. Now, does $R \in R$? If it does, then $R \notin R$, if it doesn't, then $R \in R$, contradiction.

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So, Frege's arithmetic is *inconsistent*! The discovery of Russell's paradox thus marks the end of the logicist programme.

Two Fixes

- Russell's *theory of types* (see [Russell, 1906], [Russell, 1908]): assign a type to each 'object': x^0, y^1,.. and take '$\in$' to be defined only for any two concepts x, y such that it is always the case that x has type *greater than* y or vice versa (so, $x^n \in y^{n+1}$ is permitted, $x^n \in x^n$ isn't), so Russell's paradox cannot even be formulated.

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Both 'fixes' are fraught with troubles. Russell's theory of types is cumbersome, and counterintuitive, Wright and Hale's neo-logicism does not seem to be able to definitively establish that (HP) is analytic.

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  - *Term* Formalism
  - Deductivism
  - Curry's Formalism

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- 'Generally', *inconsistency* should be avoided
- The advantage of such a position lies in the fact that it allows one to entirely avoid *metaphysical* and *epistemological* issues concerning mathematics ([Thomae, 1898])

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- (*Absence of rules*) Mathematicians seem to know what they are talking about when they utter mathematical statements even when rules are not spelt out
- (*Applicability*) What warrants the applicability of mathematics to reality, if mathematics is a *meaningless* game?
- (*Consistency*) While mathematicians might entirely disregard issues of meaning, they may want to say something about whether mathematics is consistent: this will take us well *beyond* the reach of *game formalism*

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- A possible response to (*Consistency*) is that *meta-mathematics* (which investigates consistency) is not part of the mathematical undertaking ('game')

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$$1 + 1 = 2$$

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Term Formalism

- It is the view that mathematics deals with particular signs ('terms') and their features
- Again, it is central to this view that the main goal of mathematics is to manipulate symbols
- The idea is that the *meaning* of expressions such as:

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  is that the symbols on the left-hand side of the *identity sign* can be *substituted* with the symbols lying on the right-hand side
- Term formalism is better suited to answer the objections that are raised against game formalism, for instance: (*Absence of rules*) and (*Persistence of meaning*), since it commits itself to the view that some meaning may be attached to mathematical expressions, after all.

Term Formalism: Objections and Responses

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- If such expressions are rather *types* than *tokens*, then they contain reference to *abstracta* whose existence term formalists bluntly deny

Curry's Formalism

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[Shapiro, 2000] formulates a decisive objection against Curry's position: 'what do mathematicians do when they are not operating in the context of a formalised piece of mathematics?' (p. 170).

Main Sources

For the lecture, I made extensive use of:

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- [Linnebo, 2017], chapters 2, 3, 9.

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